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The lift on a small sphere in a slow shear flow

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It is shown that a sphere moving through a very viscous liquid with velocity V relative to a uniform simple shear, the translation velocity being parallel to the streamlines and measured relative to the streamline through the centre, experiences a lift force $81\cdot 2\,\mu V a^2 \kappa^{\frac{1}{2}}/\nu^{\frac{1}{2}}$ + smaller terms perpendicular to the flow direction, which acts to deflect the particle towards the streamlines moving in the direction opposite to V. Here, a denotes the radius of the sphere, κ the magnitude of the velocity gradient, and μ and ν the viscosity and kinematic viscosity, respectively. The relevance of the result to the observations by Segré & Silberberg (1962) of small spheres in Poiseuille flow is discussed briefly. Comments are also made about the problem of a sphere in a parabolic velocity profile and the functional dependence of the lift upon the parameters is obtained.

1. Introduction

Interest in the motion of small particles carried along by Poiseuille flow through straight tubes has been stimulated in the past by observations (going back to Poiseuille 1836) that the blood corpuscles in the capillaries tend to keep away from the walls. Although Goldsmith & Mason (1962) have pointed out that the deformation of non-rigid particles will produce a lateral migration across streamlines, and Bretherton (1962*b*) has shown that rigid particles of an extreme shape may likewise migrate, the remarkable observations of Segré & Silberberg (1962), that small neutrally buoyant spheres of various sizes in Poiseuille flow through a tube slowly migrate laterally to a position distant 0·6 tube radii from the axis, have demonstrated convincingly the existence of a lateral force on rigid spherical particles.

For motion at small Reynolds number, it was pointed out by the author (1956*b*), and more fully by Bretherton (1962*b*) that no sideways force on a single rigid spherical particle can be derived on the basis of the creeping flow equations whatever the velocity profile and relative size of particle and tube, provided of course that the velocity is unidirectional. The calculation of small inertia effects is therefore relevant to an understanding of this type of phenomenon, and indeed the quantitative measurements by Segré & Silberberg of the migration rate indicate strongly that the effect is one of inertia, and not due for example to non-Newtonian effects. The full problem is one of great difficulty, as not only is the effect of inertia to be calculated for a particle in a parabolic velocity profile, but also the presence of the tube walls must be taken into account. The walls are clearly all important to the existence of the phenomenon, if only because without walls the particle would never know (so to speak) when it was the appropriate distance from the axis. However, the wall effect acts in two different ways. First, the extra drag due to the walls will make the particle lag behind the fluid; this relative velocity will be viscosity independent when the viscosity is large, and will depend only on the relative size of particle and tube and the distance from the wall. Secondly, the flow field around the particle is altered by the presence of the walls and the inertial effects will differ from those for a particle in an unbounded flow, especially when the particle is near the walls.

In the present work, we shall confine attention to the first effect of the wall and consider the force on a particle in an unbounded shear flow when the particle has a relative velocity parallel to the streamlines. This is not to say that the second effect of the wall is unimportant even when the particle is in the middle of the tube, but rather that the entire problem is too difficult and we can only deal with one aspect at the present time. Also, we shall concentrate on the case of simple shear as the force can then be calculated explicitly, but general remarks will be made about the force when the velocity profile is parabolic, and in particular the functional dependence of the force on the various parameters will be evaluated.

To be precise, we take Cartesian co-ordinates $Oxyz$, which are also labelled $Ox_1x_2x_3$ when it is convenient to use tensor notation, with origin moving with the particle so that the motion is steady. The velocity at infinity is taken to be

$$\mathbf{U} = (\kappa z + V)\,\mathbf{e}_1, \tag{1.1}$$

where $\mathbf{e}_i$ is a unit vector along Ox_i, κ is the velocity gradient, and V is the relative velocity of particle and fluid measured on the streamline through the centre. The sphere is also allowed to rotate with angular velocity $\mathbf{\Omega} = (0, \Omega, 0)$ about the y-axis. The equations of motion can be written

$$\nabla^2\mathbf{q} - \nabla p = \nu^{-1}(\mathbf{q}\,.\,\nabla)\,\mathbf{q}, \quad \nabla\,.\,\mathbf{q} = 0, \tag{1.2}$$

where $\mathbf{q} = (q_1, q_2, q_3)$ is the velocity, ν is the kinematic viscosity, the fluid is supposed incompressible, and all pressures, stresses and forces are for ease of writing 'viscosity reduced', i.e. the values given are the actual ones divided by the viscosity, unless the contrary is explicitly stated. For example, the actual pressure is p multiplied by the viscosity. The boundary conditions are

$$\mathbf{q} \to \mathbf{U} \quad \text{as} \quad r \to \infty, \qquad \mathbf{q} = \mathbf{\Omega} \wedge \mathbf{r} \quad \text{on} \quad r = a, \tag{1.3}$$

where $\mathbf{r} = (x, y, z)$ and a is the radius of the particle.

Three Reynolds numbers can be defined namely

$$R_\kappa = \kappa a^2/\nu, \quad R_V = Va/\nu, \quad R_\Omega = \Omega a^2/\nu, \tag{1.4}$$

and it is assumed that each of these is small compared with unity. As there is more than one Reynolds number, it will be convenient to think of small Reynolds number as being due to large viscosity and to obtain the effects of inertia as an expansion in inverse powers of the viscosity. The principal result of this note is that there is a lift force in the z-direction of amount

$$K V a^2 \kappa^{\frac{1}{2}}/\nu^{\frac{1}{2}} + o(\nu^{-\frac{1}{2}}) \tag{1.5}$$

for $\kappa > 0$. If $\kappa < 0$, the force is of the same magnitude but in the opposite direction. Numerical evaluation of an integral gives $K = 81{\cdot}2$.

The effect of inertia cannot be obtained by straightforward iteration of (1.2), for it is readily found like the case of a body in a uniform stream that the terms of order ν^{-1} and smaller are unable to satisfy the boundary condition at infinity. Although the technique of matching inner and outer expansions is available to overcome this problem (Kaplun & Lagerstrom 1957; Proudman & Pearson 1957), the case of bodies in shear flow is rendered difficult by the fact that fundamental solutions of the Oseen-like equation for the outer expansion are hard to obtain†. However, recent work by Childress (1964) on the question of Stokes drag in a rotating fluid has made clearer the fact that in some cases only general properties of the outer expansion are needed to determine the inner expansion to first order and that these can be obtained even though the explicit form of the outer expansion is hard to find. The present work depends entirely on this being true for a particle in a shear flow.

The plan of the work is as follows. After stating the zeroth-order term in the inner expansion, we shall find in general terms the force due to the first-order term and show that it involves two unknown constants whose values are to be determined by matching with the outer expansion. The outer expansion is then discussed and it is shown that the constants can be determined to lowest order from quite general considerations and that no detailed analysis beyond the solution of a linear first-order ordinary differential equation is called for in order to obtain the force. Finally, the relevance of the theory to the Segré & Silberberg experiment is considered, and a qualitative discussion of the force in a parabolic profile is given. In an appendix, it is pointed out that, for a spheroid freely rotating without translation in a simple shear, the first approximation to the effect of inertia may be calculated by iteration of the equations, so that an assumption made previously by the author (1956*b*) is all right.

2. The inner expansion

The solution of (1.2) can be expanded formally as

$$\mathbf{q}(\mathbf{r}) = \mathbf{q}^{(0)}(\mathbf{r}) + \mathbf{q}^{(1)}(\mathbf{r}) + \ldots, \quad p(\mathbf{r}) = p^{(0)}(\mathbf{r}) + p^{(1)}(\mathbf{r}) + \ldots, \tag{2.1}$$

successive terms being of higher order in the reciprocal of the viscosity. The zeroth-order velocity and pressure satisfy the creeping flow equations and are (see Lamb 1932, p. 596; the notation here is slightly different)

$$\mathbf{q}_0 = \mathbf{U} + \sum_n \frac{\mathbf{r}}{r^{2n+3}}[\tfrac{1}{2}r^2 p_n - (2n+1)\phi_n] - \frac{n-2}{2n(2n-1)}\frac{\nabla p_n}{r^{2n-1}} + \frac{\nabla\phi_n}{r^{2n+1}} + \frac{\nabla\chi_n \wedge \mathbf{r}}{r^{2n+1}}, \tag{2.2}$$

$$p_0 = \sum_n \frac{p_n}{r^{2n+1}}, \tag{2.3}$$

† For the two-dimensional problem of a cylinder moving in a simple shear (Bretherton 1962*a*), the basic equation for the outer expansion can be reduced to a second-order equation that describes diffusion from a line source in a shear and fundamental solutions are readily obtained by generalizing some results of Townsend (1951). In the three-dimensional problem, this simplification is not possible because the pressure cannot be eliminated by the use of a stream function, but the problem of the lift can be reduced to the solution of a single fourth-order equation, related to diffusion from a point source in a shear flow.

where

$$p_1 = -\tfrac{3}{2}Vax,\quad p_2 = -5\kappa a^3xz,\quad \phi_1 = -\tfrac{1}{4}Va^3x,\quad \phi_2 = -\tfrac{1}{2}\kappa a^5xz,\quad \chi_1 = (\Omega - \tfrac{1}{2}\kappa)\,a^3y. \tag{2.4}$$

This velocity field satisfies the boundary conditions on the sphere and at infinity. (We take it for granted that the appropriate boundary condition on the zeroth-order inner expansion at infinity is (1.3), as the correctness of this assertion can be verified from the working at a later stage.) The corresponding force on the sphere is

$$\mathbf{F}^{(0)} = 6\pi a V\mathbf{e}_1, \tag{2.5}$$

and the torque is

$$\mathbf{M}^{(0)} = 8\pi a^3(\tfrac{1}{2}\kappa - \Omega)\,\mathbf{e}_2. \tag{2.6}$$

The first-order inner expansion satisfies

$$\nabla^2\mathbf{q}^{(1)} - \nabla p^{(1)} = \nu^{-1}(\mathbf{q}^{(0)}.\nabla)\,\mathbf{q}^{(0)} = \mathbf{Q}/\nu, \text{ say};\quad \nabla.\mathbf{q}^{(1)} = 0; \tag{2.7}$$

and the boundary condition

$$\mathbf{q}^{(1)} = 0 \quad \text{on} \quad r = a. \tag{2.8}$$

The integration of (2.7) is fairly straightforward (Saffman 1956*a*), but rather laborious. Here, we are mainly interested in the force, and in particular the lift, on the sphere and much of the heavy algebra can then be by-passed. It is shown in appendix A that the force $\mathbf{F}^{(1)}$ on the particle due to $\mathbf{q}^{(1)}$, $p^{(1)}$ is given by

$$\mathbf{F}^{(1)} = -\int\frac{p^{(1)}\mathbf{r}\,dS}{R} - \int\frac{\mathbf{q}^{(1)}\,dS}{R} + R\frac{d}{dR}\int\frac{\mathbf{q}^{(1)}\,dS}{R} - \frac{1}{\nu}\int\frac{(\mathbf{r}.\mathbf{q}^{(0)})\,\mathbf{q}^{(0)}\,dS}{R}, \tag{2.9}$$

where the integrals are over any sphere $r = R$ concentric with the particle. It is also shown in appendix A that, from (2.7), the integrals regarded as functions of R satisfy the ordinary differential equations

$$R\frac{d^2}{dR^2}\int\frac{\mathbf{q}^{(1)}\,dS}{R} - \frac{d}{dR}\int\frac{p^{(1)}\,\mathbf{r}\,dS}{R} = \frac{1}{\nu}\int\mathbf{Q}\,dS, \tag{2.10}$$

$$R\frac{d^2}{dR^2}\int\frac{p^{(1)}\,\mathbf{r}\,dS}{R} - 2\frac{d}{dR}\int\frac{p^{(1)}\mathbf{r}\,dS}{R} = \frac{1}{\nu}\int\mathbf{r}(\nabla.\mathbf{Q})\,dS. \tag{2.11}$$

Now $\mathbf{Q}$ is the sum of homogeneous polynomials in x, y, z multiplied by negative integral powers of r. The right-hand sides of (2.10) and (2.11) can therefore be expressed as

$$\int\mathbf{Q}\,dS = \Sigma\mathbf{a}_nR^n,\quad \int\mathbf{r}(\nabla.\mathbf{Q})\,dS = \Sigma\mathbf{b}_nR^n. \tag{2.12}$$

Direct calculation shows that except for $\mathbf{a}_1$ and $\mathbf{b}_1$, $\mathbf{a}_n$ and $\mathbf{b}_n$ are zero for $n \geqslant -1$. Equations (2.10) and (2.11) can be integrated immediately to give

$$\int\frac{p^{(1)}\,\mathbf{r}\,dS}{R} = \frac{1}{\nu}\Sigma\frac{\mathbf{b}_nR^{n+1}}{(n+1)(n-2)} + \mathbf{A} + \mathbf{B}R^3, \tag{2.13}$$

$$\int\frac{\mathbf{q}^{(1)}\,dS}{R} = \frac{1}{\nu}\Sigma\frac{\mathbf{a}_nR^{n+1}}{n(n+1)} + \frac{1}{\nu}\Sigma\frac{\mathbf{b}_nR^{n+1}}{n(n+1)(n-2)} + \mathbf{C}R + \mathbf{D} + \tfrac{1}{2}\mathbf{B}R^3, \tag{2.14}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are unknown constant vectors. The non-vanishing of $\mathbf{a}_1$ and $\mathbf{b}_1$ implies that $\mathbf{q}^{(1)}$ is $O(r)$ for large r so that the boundary condition at infinity cannot be satisfied.

The vectors **B** and **C** are arbitrary, and come from solutions of the homogeneous equation (2.7) with $\mathbf{Q} = 0$. Indeed, **C** comes from a uniform stream of velocity $\mathbf{C}/4\pi$, and **B** is associated with a uniform pressure gradient and the associated parabolic profile. The only other term in the complementary function that can make non-zero contributions to the pressure and velocity integrals is a 'Stokeslet' with pressure field $\mathbf{s}.\mathbf{r}/r^3$, say, which would contribute $\frac{4}{3}\pi\mathbf{s}$ and $\frac{8}{3}\pi\mathbf{s}$ to (2.13) and (2.14) respectively. Thus the vectors **A** and **D** are arbitrary to the extent of a Stokeslet but they are otherwise determined for it can be shown (see appendix A) that

$$2\mathbf{A}-\mathbf{D} = \frac{1}{\nu}\int\frac{(\mathbf{r}.\mathbf{Q})\,\mathbf{r}\,dS}{R} - \frac{1}{\nu}\Sigma\frac{\mathbf{a}_n R^{n+1}}{n+1} + \frac{1}{\nu}\Sigma\frac{\mathbf{b}_n R^{n+1}}{n+1}. \tag{2.15}$$

By virtue of the boundary condition (2.8), the expression (2.14) must vanish when $r = a$, giving one relation between the arbitrary vectors. The other equations to determine them will come from the hypothesis that the inner expansion matches an outer expansion in some overlap domain. The substitution of (2.13) and (2.14) into (2.9) and evaluation for $R = a$, where the first term in (2.15) and last term of (2.9) both vanish, gives

$$\mathbf{F}^{(1)} = \frac{1}{\nu}\Sigma\left\{\frac{3}{2n}\mathbf{a}_n + \frac{3-n}{2n(n-2)}\mathbf{b}_n\right\}a^{n+1} + \tfrac{3}{4}\mathbf{B}a^3 + \tfrac{3}{2}\mathbf{C}a. \tag{2.16}$$

The z-component of the term in curly brackets has been evaluated and was found to be, after some algebra,

$$\frac{\pi V\Omega a^3}{\nu} - \frac{11}{8}\frac{\pi V\kappa a^3}{\nu}. \tag{2.17}$$

But in order to find the lift, it is also necessary to determine the z-component of **B** and **C**, and we shall now consider how these may be found. It will turn out that in fact **C** is $O(\nu^{-\frac{1}{2}})$ and is the dominant term. For future reference, it should be stressed that **C** is determined by the part of the inner expansion which matches a uniform stream at infinity. It should also be noticed that there are no logarithmic terms in the first-order inner expansion for our problem.

3. The outer expansion for simple shear

We can imagine the body removed and replaced by a distribution of body forces $\mathbf{f}(\mathbf{r})$ in the region $r \leqslant a$, where the force field **f** has to be chosen so that $\mathbf{q} = 0$ on the surface $r = a$. Since no fluid crosses the boundary, two sets of forces will produce the same motion outside $r = a$ if their moments of all orders are the same. In particular, the body may be replaced by point forces, couples, etc., at the origin provided the moments of the point forces, etc., are equal to the moments of the actual surface stresses on the body. The equation of motion can now be written as

$$\nabla p - \nabla^2\mathbf{q} + \frac{1}{\nu}(\mathbf{q}.\nabla)\,\mathbf{q} = \mathbf{f} = {}_1\mathbf{P}\delta(\mathbf{r}) + {}_2\mathbf{P}.\nabla\delta(\mathbf{r}) + {}_3\mathbf{P}.\nabla\nabla\delta(\mathbf{r}) + \ldots,\dagger \tag{3.1}$$

† We employ the notation that ${}_n\mathbf{P}.\mathbf{aa}\ldots\mathbf{a}$, where **a** occurs $n-1$ times, is a vector with i-component $P_{ijk\ldots n}a_j a_k \ldots a_n$.

where ${}_n\mathbf{P}$ is a nth order tensor, $\delta(\mathbf{r})$ is the three-dimensional delta function, and

$$ {}_n\mathbf{P} = (-1)^{n-1}\int \mathbf{f}\mathbf{r}\mathbf{r}\ldots\mathbf{r}\,dV, \tag{3.2}$$

$\mathbf{r}$ occurring $n-1$ times in the integrand.

To zero order in the viscosity, i.e. with the force as given by the creeping flow equations,

$$ {}_1\mathbf{P} = -6\pi a V\mathbf{e}_1, \tag{3.3}$$

$$ {}_2\mathbf{P} = 4\pi a^3(\Omega-\tfrac{1}{2}\kappa)\begin{pmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix} - \tfrac{10}{3}\pi a^3\kappa\begin{pmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}, \tag{3.4}$$

and so on, the odd-order tensors involving V and the even order ones κ and Ω. The antisymmetrical part of ${}_2\mathbf{P}$ is the couple $8\pi a^3(\frac{1}{2}\kappa-\Omega)$ about the y-axis.

In the usual way for the determination of outer expansions, we introduce a strained co-ordinate

$$ \tilde{\mathbf{r}} = S\mathbf{r}, \tag{3.5}$$

where $S \to 0$ as $\nu \to \infty$, and write

$$ \mathbf{q} = \mathbf{U} + \mathbf{q}'(\tilde{\mathbf{r}}), \tag{3.6}$$

where $\mathbf{q}' \to 0$ as $\tilde{\mathbf{r}} \to \infty$ and $\mathbf{q}' \to 0$ as $\nu \to \infty$ for $\tilde{\mathbf{r}}$ fixed. In terms of the strained co-ordinate, equation (3.1) becomes

$$ \nu S\tilde{\nabla}p - \nu S^2\tilde{\nabla}^2 q' + \kappa(\tilde{z}\,\partial\mathbf{q}'/\partial\tilde{x} + q_3'\mathbf{e}_1) + VS\,\partial\mathbf{q}'/\partial\tilde{x} + S(\mathbf{q}'.\tilde{\nabla})\,\mathbf{q}' \\ = \nu S^3\,{}_1\mathbf{P}\delta(\tilde{\mathbf{r}}) + \nu S^4\,{}_2\mathbf{P}.\tilde{\nabla}\,\delta(\tilde{\mathbf{r}}) + \nu S^5\,{}_3\mathbf{P}.\tilde{\nabla}\tilde{\nabla}\,\delta(\tilde{\mathbf{r}}) + \ldots, \tag{3.7}$$

after multiplication throughout by ν. The equation of continuity remains

$$ \tilde{\nabla}.\mathbf{q}' = 0. \tag{3.8}$$

The straining factor is determined by the requirement that the viscous and inertia terms be of comparable order, from which it follows that

$$ \nu S^2 = 1, \quad S = \nu^{-\frac{1}{2}}. \tag{3.9}$$

Then to lowest order, i.e. in the limit $\nu \to \infty$ with $\tilde{\mathbf{r}}$ fixed, we see that $\mathbf{q}'$ satisfies the equation

$$ \tilde{\nabla}p' - \tilde{\nabla}^2\mathbf{q}' + \kappa(\tilde{z}\,\partial\mathbf{q}'/\partial\tilde{x} + q_3'\mathbf{e}_1) = -(6\pi a V/\nu^{\frac{1}{2}})\,\mathbf{e}_1\delta(\tilde{\mathbf{r}}), \tag{3.10}$$

where $p' = \nu^{\frac{1}{2}}p$ and we have used the fact that to lowest order the force on the particle is the Stokes drag. (The right-hand side of (3.10) can also be obtained by remarking that the outer expansion is equivalent to shrinking the body to a point, so that to lowest order the body is equivalent to a point-force of magnitude equal to the drag.) According to (3.10), $\mathbf{q}'$ is $O(\nu^{-\frac{1}{2}})$ and inspection of (3.7) implies heuristically that the error in using (3.10) is $O(\nu^{-1})$.

Before we solve (3.10), an intuitive discussion would appear to be useful. The velocity $\mathbf{q}'$ is analytic except at $\tilde{r} = 0$, where it has the behaviour [obtained by retaining only the highest-order derivatives in (3.10) and using $\nabla^2(1/\tilde{r}) = -4\pi\delta(\tilde{\mathbf{r}})$]

$$ \mathbf{q}' \sim -\frac{3}{4}\frac{aV}{\nu^{\frac{1}{2}}}\left(\frac{\mathbf{e}_1}{\tilde{r}} + \frac{(\mathbf{e}_1.\tilde{\mathbf{r}})\,\tilde{\mathbf{r}}}{\tilde{r}^3}\right) = \mathbf{q}'_s, \quad \text{say.} \tag{3.11}$$

This is the velocity field in strained co-ordinates of a Stokeslet. Furthermore, the equation is elliptic (the elimination of the pressure by means of the continuity equation gives an equation in which the highest derivatives are of the biharmonic form) with coefficients that are polynomials in $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$. The solution will be expressible for $\tilde{r} < \infty$ as a series of polynomials in $\tilde{x}$, $\tilde{y}$, $\tilde{z}$, each being multiplied by a power of $\tilde{r}$ (and possibly also powers of $\log \tilde{r}$). Symbolically, we may write

$$\mathbf{q}' = \frac{aV\kappa^{\frac{1}{2}}}{\nu^{\frac{1}{2}}}\left\{\frac{\mathbf{H}^{(0)}(\kappa^{\frac{1}{2}}\tilde{\mathbf{r}})}{\kappa^{\frac{1}{2}}\tilde{r}} + \mathbf{H}^{(1)}(\kappa^{\frac{1}{2}}\tilde{\mathbf{r}}) + \kappa^{\frac{1}{2}}\tilde{r}\mathbf{H}^{(2)}(\kappa^{\frac{1}{2}}\tilde{\mathbf{r}}) + \kappa\tilde{r}^2\mathbf{H}^{(3)}(\kappa^{\frac{1}{2}}\tilde{\mathbf{r}}) + \ldots\right\}, \quad (3.12)$$

where the $\mathbf{H}$ are homogeneous functions of degree zero in $\kappa^{\frac{1}{2}}\tilde{x}$, $\kappa^{\frac{1}{2}}\tilde{y}$, $\kappa^{\frac{1}{2}}\tilde{z}$, and $\mathbf{H}^{(0)}$ is given by (3.11). The dependence on $\kappa^{\frac{1}{2}}$ follows from the fact that $\tilde{r}$ has the dimensions of $\kappa^{-\frac{1}{2}}$ and the solution of (3.10) is linearly proportional to $aV/\nu^{\frac{1}{2}}$. For definiteness, and without loss of generality, we take $\kappa > 0$. If $\kappa < 0$, we simply reverse the y- and z-axes. The $\mathbf{H}$ are determined completely in principle by (3.10) and the boundary condition $\mathbf{q}' \to 0$ as $\tilde{r} \to \infty$, although their actual calculation is a matter of great difficulty.

The matching technique to be employed now relies on the hypothesis that the outer expansion near $\tilde{r} = 0$ fits the inner expansion (see § 2) in the vicinity of $r = \infty$. Writing (3.12) in unstrained co-ordinates, we have

$$\mathbf{q}' = aV\left\{\frac{\mathbf{H}^{(0)}}{r} + \frac{\kappa^{\frac{1}{2}}}{\nu^{\frac{1}{2}}}\mathbf{H}^{(1)} + \frac{\kappa r}{\nu}\mathbf{H}^{(2)} + \frac{\kappa r^2}{\nu^{\frac{3}{2}}}\mathbf{H}^{(3)} + \ldots\right\}, \quad (3.13)$$

and the terms $O(\nu^{-1})$ or larger must agree with those in the first two terms of the inner expansion as $r \to \infty$. Note that since the $\mathbf{H}$ are homogeneous of degree zero, they are in fact functions not of the argument $\kappa^{\frac{1}{2}}\mathbf{r}/\nu^{\frac{1}{2}}$ but only of the direction $\mathbf{r}$ and are independent of ν.

Neglecting terms smaller than $O(\nu^{-1})$, we have from (3.13) that

$$\int \frac{\mathbf{q}'\,dS}{R} = aV\int\frac{\mathbf{H}^{(0)}\,dS}{R^2} + \frac{aV\kappa^{\frac{1}{2}}}{\nu^{\frac{1}{2}}}R\int\frac{\mathbf{H}^{(1)}\,dS}{R^2} + \frac{aV\kappa}{\nu}R^2\int\frac{\mathbf{H}^{(2)}\,dS}{R^2}, \quad (3.14)$$

where the integrals are numbers. The first term, which is independent of ν, matches the Stokeslet in the zeroth-order inner expansion. The second term, being $O(\nu^{-\frac{1}{2}})$, must match terms from the first-order inner expansion, and comparison with (2.14) shows that to $O(\nu^{-\frac{1}{2}})$

$$\mathbf{C} = \frac{aV\kappa^{\frac{1}{2}}}{\nu^{\frac{1}{2}}}\int\frac{\mathbf{H}^{(1)}\,dS}{R^2}, \quad \mathbf{B} = 0. \quad (3.15)$$

The third term in (3.14) will match the contribution from $\mathbf{a}_1$ and $\mathbf{b}_1$ in (2.14). It has been implicitly assumed that there are no logarithmic terms in (3.13) to $O(\nu^{-1})$ because there are no such terms in (2.14) for them to match with. It is known that they occur to higher order in the drag, and this is also probably true for the lift-force in a shear flow. It should be stressed that (3.14) does not determine $\mathbf{C}$ to order ν^{-1} as (3.12) is in error to this order. For work to this accuracy, it would be necessary to retain further terms in the equation (3.10). The expression (2.17) does not therefore give all the terms that are $O(\nu^{-1})$.

It now follows from (2.16) that the (viscosity reduced) force on the particle is

$$6\pi a V \mathbf{e}_1 + \tfrac{3}{2} V \frac{a^2 \kappa^{\frac{1}{2}}}{\nu^{\frac{1}{2}}} \int \frac{\mathbf{H}^{(1)} dS}{R^2} + O(\nu^{-1}), \tag{3.16}$$

where the second term is to be found by solving (3.10) and forming the expansion (3.12). In the next section, we shall show how this can be done for the lift component without obtaining a complete solution. It can be said that the object of the calculation is to determine the part of the flow in the neighbourhood of $\tilde{r} = 0$ which is analytic there and which appears to the inner expansion as a uniform translation of the fluid at infinity. It is the average value of $\mathbf{q}'$ throughout a small sphere centred on $\tilde{r} = 0$.

4. The lift

We introduce the three-dimensional Fourier transform $\mathbf{\Gamma}(k) = (\Gamma_1, \Gamma_2, \Gamma_3)$ of the velocity field, defined by†

$$\mathbf{\Gamma}(\mathbf{k}) = \frac{1}{8\pi^3} \int \mathbf{q}' e^{-i\mathbf{k}\cdot\mathbf{r}} d\mathbf{r}, \quad \mathbf{q}'(\mathbf{r}) = \int \mathbf{\Gamma}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} d\mathbf{k}. \tag{4.1}$$

In Fourier space, equation (3.10) becomes

$$\kappa\{-k_1(\partial \mathbf{\Gamma}/\partial k_3) + \Gamma_3 \mathbf{e}_1\} - i\mathbf{k}\Pi + k^2 \mathbf{\Gamma} = -(3aV/4\pi^2\nu^{\frac{1}{2}})\,\mathbf{e}_1, \tag{4.2}$$

where $\Pi(\mathbf{k})$ is the transform of p' and $\mathbf{k} = (k_1, k_2, k_3)$. For the lift or transverse force in the z-direction, it is sufficient to determine q_3' or equivalently Γ_3. The transform of the continuity equation is

$$\mathbf{k}.\mathbf{\Gamma} = 0. \tag{4.3}$$

Taking the scalar product of (4.2) with $\mathbf{k}$ using (4.3), and noting that

$$\mathbf{k}.\partial\mathbf{\Gamma}/\partial k_3 = -\Gamma_3,$$

we obtain
$$-ik^2\Pi = -(3Va/4\pi^2\nu^{\frac{1}{2}})\,k_1 - 2\kappa k_1 \Gamma_3, \tag{4.4}$$
where $k = |\mathbf{k}|$.

It follows immediately from (4.2) that Γ_3 satisfies

$$\kappa\left(k_1 \frac{\partial \Gamma_3}{\partial k_3} + 2\frac{k_1 k_3}{k^2}\Gamma_3\right) - k^2 \Gamma_3 = \frac{-3Va}{4\pi^2 \nu^{\frac{1}{2}}} \frac{k_1 k_3}{k^2}. \tag{4.5}$$

The solution of (3.10) must have the Stokeslet singularity (3.11) at $\tilde{r} = 0$ but otherwise should be finite and tend to zero as $\tilde{r} \to \infty$. The Fourier transform of the Stokeslet is most easily obtained by letting $\kappa \to 0$ in (4.5). The condition that $q_3' - q_{s3}'$ is bounded is equivalent to

$$\Gamma_3 - \frac{3Va}{4\pi^2\nu^{\frac{1}{2}}} \frac{k_1 k_3}{k^4} \tag{4.6}$$

being integrable over the entire $\mathbf{k}$-space.

† This technique seems to have been used first in problems of this type by Childress (1964).

Equation (4.5) is readily integrated and gives

$$k^2\Gamma_3 \exp\left\{\frac{(k_1^2+k_2^2)\,k_3}{\kappa k_1}-\frac{1}{3}\frac{k_3^2}{\kappa k_1}\right\} = -\frac{3Va}{4\pi^2\kappa\nu^{\frac{1}{2}}}\int^{k_3} u\exp\left\{-\frac{(k_1^2+k_2^2)\,u}{\kappa k_1}-\frac{1}{3}\frac{u^3}{\kappa k_1}\right\}du. \quad (4.7)$$

In order that Γ_3 should vanish as $k \to \infty$, the bottom limit of integration should be $+\infty$ for $k_1 > 0$ and $-\infty$ for $k_1 < 0$. It is convenient to change variable and write

$$\xi = (u-k_3)/k_1, \quad (4.8)$$

whence

$$\Gamma_3 = \frac{3Va}{4\pi^2\kappa\nu^{\frac{1}{2}}}\int_0^\infty \frac{\xi k_1^2+k_1k_3}{k^2}\exp\left\{-\frac{1}{\kappa}(\xi k^2+\xi^2 k_1 k_3+\tfrac{1}{3}\xi^3 k_1^2\right\}d\xi. \quad (4.9)$$

It is easy to show that the expression (4.6) is then integrable. The inversion formula (4.1) now gives q_3' as a Fourier integral, from which we can form an expansion like (3.12) about $\tilde{r} = 0$. The term independent of $\tilde{r}$ in this expansion is the value of $q_3' - q_{s3}'$ at $\tilde{r} = 0$, namely

$$\int\left(\Gamma_3+\frac{3Va}{4\pi^2\nu^{\frac{1}{2}}}\frac{k_1k_3}{k^4}\right)d\mathbf{k} = -\frac{aV\kappa^{\frac{1}{2}}}{\nu^{\frac{1}{2}}}H_3^{(1)} = -\frac{aV\kappa^{\frac{1}{2}}}{4\pi\nu^{\frac{1}{2}}}\int\frac{H_3^{(1)}dS}{R^2} \quad (4.10)$$

since $H_3^{(1)}$ is independent of the direction of $\tilde{\mathbf{r}}$. Hence from (3.16), the lift force L in the positive z-direction is

$$\begin{aligned} L &= 6\pi a\int\left(\Gamma_3-\frac{3Va}{4\pi^2\nu^{\frac{1}{2}}}\frac{k_1k_3}{k^4}\right)dk \\ &= \frac{9\pi Va^2\kappa^{\frac{1}{2}}}{\nu^{\frac{1}{2}}}\frac{(\frac{1}{4})!}{(\frac{3}{4})!}\int_0^\infty\int_0^\infty\frac{(3t^2+\frac{1}{4}\eta^2+1)\,dt\,d\eta}{\eta^{\frac{1}{2}}(t^2+\frac{1}{12}\eta^2+1)^{\frac{3}{4}}\,[(t^2+\frac{1}{4}\eta^2+1)^2-\eta^2t^2]} \\ &= KVa^2\kappa^{\frac{1}{2}}/\nu^{\frac{1}{2}}, \quad \text{say}, \end{aligned} \quad (4.11)$$

after substituting (4.9), integrating with respect to the magnitude of **k**, and re-writing. The value of K has been obtained by numerical integration and is 81·2†. It will be remembered that the actual force is μL. To lowest order, the lift force would produce a transverse velocity W given by

$$W = L/6\pi a = KVa\kappa^{\frac{1}{2}}/6\pi\nu^{\frac{1}{2}}, \quad (4.12)$$

if the particle were free to move.

The result (4.11) may be compared with that found by Bretherton (1962*a*, equation (10))‡, which with the present notation gives a lift force per unit length on a circular cylinder of radius a of about

$$74{\cdot}5\,V\,(\log\kappa a^2/\nu)^{-2}+O(\log\kappa a^2/\nu)^{-3}. \quad (4.13)$$

If V is very large, the result (4.11) will become invalid, essentially because the step from (3.7) to (3.10) requires that $V/(\nu\kappa)^{\frac{1}{2}} \ll 1$. Thus the analysis does not give the lift on a sphere moving through a fluid when there is a small shear with

† The double integral was reduced to a single integral by a series of complicated transformations, which was kindly evaluated numerically by Dr Klaus Jacob of the Booth Computing Centre at Caltech.

‡ There is a misprint in this equation, a minus sign being omitted before the second term.

$\kappa \ll \nu/V^2$; this is another problem, which must be solved in a different way. Sufficient conditions for the validity of (4.12) may be written as

$$R_V \ll R_\kappa^{\frac{1}{2}}, \quad R_\kappa \ll 1, \quad R_\Omega \ll 1. \tag{4.14}$$

It is interesting that to $O(\nu^{-\frac{1}{2}})$, the lift force is independent of the rotation of the particle. Indeed, a further inspection of equation (3.7) for the outer expansion shows that the rotation enters into the lift through the integration of the term involving V on the left-hand side and the term containing ${}_2\mathbf{P}$ on the right-hand side, since by simple symmetry arguments the interaction of simple shear and particle rotation cannot produce a force but only a couple. Since each of these terms is smaller by a factor $\nu^{-\frac{1}{2}}$ than those retained in (3.10), the outer expansion contributes a term $O(\nu^{-\frac{3}{2}})$ to the lift force which is smaller than

$$L = \pi V \Omega a^3/\nu, \tag{4.15}$$

as given by the iterated inner expansion (2.17). The result (4.15) was given by Rubinow & Keller (1961) and applied by them to the Segré & Silberberg phenomenon, but the present work shows that unless the rotation speed is very much greater than the rate of shear, and for a freely rotating particle $\Omega = \frac{1}{2}\kappa$, the lift force due to particle rotation is less by an order of magnitude than that due to the shear when the Reynolds number is small.

It is worth mentioning that the equation for Γ_1 (and Γ_2) can also be integrated and Γ_1 expressed as an integral of Γ_3 and known functions. The extra drag proportional to $Va^2\kappa^{\frac{1}{2}}/\nu^{\frac{1}{2}}$ can then be written as an integral like (4.11), but the expression is considerably more complicated and at the present stage the result does not appear to justify the apparently great labour of computation.

5. Discussion

Let us now consider the extent to which the previous analysis may apply to the motion of a single particle in Poiseuille flow through a tube of radius b. The velocity profile in the tube is

$$u = 2u_m(1-\rho^2/b^2), \tag{5.1}$$

where u_m is the mean velocity and ρ the distance from the axis. The tube Reynolds number Re is defined as $2u_m b/\nu$. A particle of radius a whose centre is at station ρ sees locally a shear of magnitude $4u_m\rho/b^2$. If, therefore, the particle lags behind the fluid with relative velocity V, measured relative to the streamline through the centre, the result (4.11) implies a (viscosity reduced) force

$$2KVa^2(u_m\rho/\nu b^2)^{\frac{1}{2}} \text{ towards the axis,} \tag{5.2}$$

and an equivalent inwards velocity

$$\frac{KV}{3\pi}\left(\frac{u_m\rho a^2}{\nu b^2}\right)^{\frac{1}{2}}. \tag{5.3}$$

If, on the other hand, the particle goes faster than the fluid, the inertial effect will move the particle away from the axis of the tube.

Three conditions must be satisfied before (5.2) or (5.3) can be thought relevant. First, the neglect of profile curvature must be reasonable, and an estimate for this condition to be valid can be obtained as follows. According to the general ideas of inner and outer expansions and the details of § 3, the domain of the outer expansion is $\kappa^{\frac{1}{2}}r/\nu^{\frac{1}{2}} > 1$, and we require that the profile curvature be negligible over a sizable portion of the outer domain. In other words, the departure from linearity of the profile over a region of linear dimension $\nu^{\frac{1}{2}}/\kappa^{\frac{1}{2}}$ should be small, which gives from (5.1)

$$\rho\nu^{\frac{1}{2}}/\kappa^{\frac{1}{2}} \gg \nu/\kappa, \quad \text{i.e.} \quad \rho(\kappa/\nu)^{\frac{1}{2}} = 2^{\frac{1}{2}}(\rho/b)^{\frac{3}{2}}Re^{\frac{1}{2}} \gg 1. \tag{5.4}$$

Thus the tube Reynolds number must be large compared with unity and the particle not too close to the axis. Secondly, for the wall effect to be negligible except in so far as it determines V, the wall must lie well inside the outer domain which implies

$$\frac{b-\rho}{b} \gg \frac{\nu^{\frac{1}{2}}}{b\kappa^{\frac{1}{2}}} = \frac{1}{2}\left(\frac{\nu}{u_m\rho}\right)^{\frac{1}{2}} = \frac{1}{2^{\frac{1}{2}}}\left(\frac{b}{\rho}\right)^{\frac{1}{2}}\frac{1}{Re^{\frac{1}{2}}}. \tag{5.5}$$

If (5.4) is satisfied, the right-hand side of (5.5) will be small compared with unity. And thirdly, the particle Reynolds number must be small, i.e.

$$\frac{\kappa^{\frac{1}{2}}a}{\nu^{\frac{1}{2}}} = 2^{\frac{1}{2}}\frac{a}{b}\left(\frac{\rho}{b}\right)^{\frac{1}{2}}Re^{\frac{1}{2}} \ll 1. \tag{5.6}$$

The first and third conditions are somewhat restrictive, as they imply large tube Reynolds numbers and very small particles, and hence it turns out that quantitative application to the Segré & Silberberg effect is limited. However the results (5.2) and (5.3) are in qualitative agreement with some observations by Oliver (1962), who watched particles in Poiseuille flow through a vertical tube with relative velocities produced by buoyancy effect. He found that downward settling particles in downward flowing liquid moved towards the wall, and upwards settling particles moved towards the axis, in agreement with (5.2) and (5.3). Although the condition (5.4) was satisfied in these experiments, the value of a/b was about $\frac{1}{4}$ so that (5.6) was violated except for particles very close to the axis and the theory is not strictly applicable, but the agreement is encouraging. On the other hand, Oliver also found that the effect of particle rotation, which was stopped by using eccentrically weighted spheres, was the opposite of that predicted by (4.15).

Quantitative comparison with the Segré & Silberberg results is not possible because the relative velocity V is unknown, and there is no obvious way of calculating it. Moreover, most of their experiments were at lower values of the Reynolds number than permitted by (5.4). However, one feature of their experiments is consistent with the present theory and this is a systematic decrease in the observed annulus radius for $Re > 30$. Although this might be connected with entry-length phenomenon, it is interesting that (5.4) predicts that the phenomenon might start to change when Re is around 10, and also the sign of the change is in accord with the theory.

It is now appropriate to inquire about the force on a particle in a parabolic velocity profile

$$\mathbf{U} = \{\alpha(y^2+z^2)+\kappa z+V\}\,\mathbf{e}_1, \tag{5.7}$$

where without loss of generality it may be supposed that $\alpha > 0$. An inner expansion can be constructed as in §2, but the force cannot be found without some knowledge of the outer expansion. At the present time, no way has been discovered of overcoming the analytical difficulties involved, but it is possible to say something about the dependence of the lift force on the various parameters, although the magnitude and sign of the coefficients cannot be determined. We make the substitution (3.6) with the flow at infinity given by (5.7) but now we introduce the strained co-ordinates

$$\mathbf{r}^* = (\alpha^{\frac{1}{3}}/\nu^{\frac{1}{3}})\,\mathbf{r}. \tag{5.8}$$

Note that $\mathbf{r}^*$ is dimensionless. The equation for $\mathbf{q}'$ now becomes

$$\begin{aligned}\nabla^* p^* - \nabla^{*2}\mathbf{q}' &+ \left\{(y^{*2}+z^{*2})\frac{\partial \mathbf{q}'}{\partial x^*}+2(y^* q_2'+z^* q_3')\,\mathbf{e}_1\right\} \\ &+\frac{\kappa}{\alpha^{\frac{2}{3}}\nu^{\frac{1}{3}}}\left(z^*\frac{\partial \mathbf{q}'}{\partial x^*}+q_3'\mathbf{e}_1\right)+\frac{V}{\alpha^{\frac{1}{3}}\nu^{\frac{2}{3}}}\frac{\partial \mathbf{q}'}{\partial x^*}+\frac{1}{\alpha^{\frac{1}{3}}\nu^{\frac{2}{3}}}(\mathbf{q}'.\nabla^*)\,\mathbf{q}' \\ &= (\alpha/\nu)^{\frac{1}{3}}\,{}_1\mathbf{P}\,\delta(\mathbf{r}^*)+(\alpha/\nu)^{\frac{2}{3}}\,{}_2\mathbf{P}.\nabla^*\delta(\mathbf{r}^*)+\ldots,\end{aligned} \tag{5.9}$$

where now the zeroth-order approximation gives (see Simha 1936)

$${}_1\mathbf{P} = -6\pi a(V+\tfrac{2}{3}\alpha a^2)\,\mathbf{e}_1, \tag{5.10}$$

and ${}_2\mathbf{P}$ is still given by (3.4). As discussed in §4, the lift force is proportional to a multiplied by the finite residue of q_3' at $\mathbf{r}^* = 0$ after the singularities have been subtracted out. By symmetry, the lift must involve α or V and κ or Ω, since there will be no lift on a particle in a symmetrical flow field. Hence the contribution to the finite residue of q_3', which comes from ${}_1\mathbf{P}$ and the dominant terms on the left-hand side of (5.9) that do not contain ν, and which is $O(\nu^{-\frac{1}{3}})$ if $V+\frac{2}{3}\alpha a^2 \neq 0$, does not contribute to the lift. For

$$\kappa/\alpha^{\frac{2}{3}}\nu^{\frac{1}{3}} \ll 1, \quad \text{i.e.} \quad 2(\rho/b)\,Re^{\frac{1}{3}} \ll 1, \tag{5.11}$$

the solution of (5.9) will be expressible as a series in $\kappa/\alpha^{\frac{2}{3}}\nu^{\frac{1}{3}}$ for finite r^*, and the leading terms in the lift force will be proportional to

$$\left.\begin{aligned}&\frac{a\kappa}{\alpha^{\frac{2}{3}}\nu^{\frac{1}{3}}}\left(\frac{\alpha}{\nu}\right)^{\frac{1}{3}}a(V+\tfrac{2}{3}\alpha a^2) = \frac{a^2\kappa}{\alpha^{\frac{1}{3}}\nu^{\frac{2}{3}}}(V+\tfrac{2}{3}\alpha a^2),\\ \text{and}\quad &a(\alpha/\nu)^{\frac{2}{3}}a^3\kappa = a^4\kappa(\alpha/\nu)^{\frac{2}{3}},\quad a(\alpha/\nu)^{\frac{2}{3}}a^3\Omega = a^4\Omega(\alpha/\nu)^{\frac{2}{3}}.\end{aligned}\right\} \tag{5.12}$$

Note that the condition (5.11) is the converse of (5.4), which can be written as $\alpha\nu^{\frac{1}{2}}/\kappa^{\frac{1}{2}} \ll \kappa$, so that (5.12) may be expected to apply for tube Reynolds numbers less than unity. The drift velocities corresponding to the forces (5.12) would be proportional to

$$(a\kappa/\alpha^{\frac{1}{3}}\nu^{\frac{2}{3}})\,(V+\tfrac{2}{3}\alpha a^2), \quad a^3\kappa(\alpha/\nu)^{\frac{2}{3}}, \quad a^3\Omega(\alpha/\nu)^{\frac{2}{3}}. \tag{5.13}$$

It is interesting that the last two expressions give a velocity proportional to the third power of the particle radius, as was in fact observed by Segré & Silberberg.

It remains to determine the coefficients in (5.13) and the value of $V + \frac{2}{3}\alpha a^2$, but this problem has so far proved intractable. The results (5.13) also predict by the dependence on viscosity that the drift velocity should be proportional to $Re^{\frac{2}{3}}$ if V is independent of ν. Segré & Silberberg found that a reasonable correlation existed for the data if the transverse velocity was assumed linearly proportional to Re, but the completeness of the data is not such as to rule out a $Re^{\frac{2}{3}}$-dependence. The result (5.3) valid for large tube Reynolds numbers gives a $Re^{\frac{1}{2}}$-behaviour.

In a previous attempt to find the lift on a sphere in a parabolic velocity profile, the author (1956*a*, *b*) iterated the Navier-Stokes equations assuming in effect that B_3 and C_3 (see equation (2.16)) were zero. This procedure gives a force proportional to $-\alpha\kappa a^4/\nu$. The present work does not invalidate the actual calculation, and there is a term of this order in the force, but it is now clear that there are larger terms of order $\nu^{-\frac{2}{3}}$ generated by the matching of inner and outer expansions. The author's (1956*b*) paper was devoted primarily to the effect of inertia upon the rotation of a spheroid in a uniform shear and the systematic change in orbit due to inertia effects was calculated by iteration, also on the assumption that matching with an outer expansion would have no effect to lowest order. It is shown in appendix B that this last assumption is in fact valid, so that the present work does not lead to a reappraisal of the main conclusions of the previous paper.

This work was started while acting as a vacation consultant at the National Physical Laboratory, Teddington. I am grateful to Dr J. T. Stuart for reawakening my interest in the subject.

Appendix A

Let
$$p_{ij} = -p + \left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i}\right) \tag{A 1}$$
denote the components of the (viscosity reduced) stress tensor due to the first-order inner expansion. For convenience, the superscript 1 is dropped. The components satisfy the equation
$$\frac{\partial p_{ij}}{\partial x_j} = \frac{1}{\nu}\frac{\partial(q_i^{(0)}q_j^{(0)})}{\partial x_j} = \frac{Q_i}{\nu}. \tag{A 2}$$
The force on the particle due to these stresses is
$$F_i = \int_{r=a} p_{ij}n_j dS = \int_{r=R} p_{ij}n_j dS - \frac{1}{\nu}\int_{r=R} q_i^{(0)}q_j^{(0)}n_j dS, \tag{A 3}$$
where R is the radius of any sphere surrounding and concentric with the particle and $\mathbf{n}$ is the outwards normal.

We require the following result. For any function ϕ,
$$\int_{r=R}\frac{\partial\phi}{\partial x_i}dS = \frac{d}{dR}\int_{r\leqslant R}\frac{\partial\phi}{\partial x_i}dV = \frac{d}{dR}\int_{r=R}\frac{\phi x_i}{R}dS, \tag{A 4}$$
where the first step states that the surface integral is simply the derivative of the volume integral with respect to the radius of the bounding surface, and the

second step is a consequence of the divergence theorem and the fact that $n_i = x_i/R$ when the origin is at the centre of the sphere.

Then
$$\int \left(\frac{\partial q_i}{\partial x_j}+\frac{\partial q_j}{\partial x_i}\right)\frac{x_j}{R}dS = \frac{1}{R}\int\left[\frac{\partial(q_i x_j)}{\partial x_j}+\frac{\partial(q_j x_j)}{\partial x_i}-4q_i\right]dS$$
$$= \frac{1}{R}\frac{d}{dR}\int\frac{q_i x_j^2 dS}{R}+\frac{1}{R}\frac{d}{dR}\int\frac{q_j x_j x_i dS}{R}-4\int\frac{q_i dS}{R}$$
$$= -2\int\frac{q_i dS}{R}+R\frac{d}{dR}\int\frac{q_i dS}{R}+\frac{1}{R}\int\frac{\partial(x_i q_j)}{\partial x_j}dS$$
$$= -\int\frac{q_i dS}{R}+R\frac{d}{dR}\int\frac{q_i dS}{R}, \tag{A 5}$$

where we have used $x_j^2 = R$ and the continuity equation $\partial q_j/\partial x_j = 0$, and all integrals are over $r = R$. The substitution of (A 1) into (A 3) and use of (A5) gives equation (2.9).

To derive equation (2.10), we integrate (2.7) over the sphere $r = R$, use the result (A 4), and the further result that for any function ϕ

$$\int \nabla^2\phi dS = \frac{d}{dR}\int\frac{\partial\phi}{\partial x_i}\frac{x_i}{R}dS = \frac{d}{dR}\int\frac{1}{R}\left\{\frac{\partial(\phi x_i)}{\partial x_i}-3\phi\right\}dS$$
$$= \left(\frac{d^2}{dR^2}-\frac{2}{R}\frac{d}{dR}+\frac{2}{R^2}\right)\int\phi dS = R\frac{d^2}{dR^2}\int\frac{\phi dS}{R} \tag{A 6}$$

on repeated applications of (A 4).

To obtain equation (2.11), we take the divergence of (2.7), multiply by $\mathbf{r}$, and integrate over the sphere $r = R$. The equation follows from use of the identity

$$x_i\nabla^2\phi = \nabla^2(x_i\phi)-2\partial\phi/\partial x_i \tag{A 7}$$

and (A 4) and (A 6).

The derivation of equation (2.15) is as follows. Repeated application of (A 4), (A 6) and (A 7) gives

$$\int x_i x_j \nabla^2 q_j dS = R^2\frac{d}{dR}\int\frac{q_i dS}{R}-R\int\frac{q_i dS}{R}, \tag{A 8}$$

$$\int x_i x_j \frac{\partial p}{\partial x_j}dS = R^2\frac{d}{dR}\int\frac{p x_i dS}{R}-2R\int\frac{p x_i dS}{R}. \tag{A 9}$$

Then from equation (2.7),

$$\frac{1}{\nu}\int x_i x_j Q_j dS = \int x_i x_j\left(\nabla^2 q_j-\frac{\partial p}{\partial x_j}\right)dS,$$

and (2.15) follows on using (A 8) and (A 9), substituting (2.13) and (2.14), and carrying out the reduction.

Appendix B

Let us consider a spheroidal particle freely rotating in a simple shear $q_1 = \kappa z$ In general the motion is unsteady, and the equation of motion is

$$\nabla^2 \mathbf{q} - \nabla p = \frac{1}{\nu}(\mathbf{q}.\nabla)\,\mathbf{q} + \frac{1}{\nu}\frac{\partial \mathbf{q}}{\partial t}. \tag{B 1}$$

At infinity, the velocity perturbation for the zeroth-order solution is proportional to $a^3\kappa/r^2$, where a now denotes a typical dimension of the body. We suppose that the body moves so that there is no couple on it due to the Stokes flow. An inner expansion like (2.1) can then be constructed and the small couple due to inertia can be calculated (Saffman 1956*b*). However, the first-order term in the inner expansion of the velocity field is proportional to $a^3\kappa^2/\nu$ as $r \to \infty$, which does not vanish and hence the boundary condition at infinity is violated.

We neglect buoyancy forces and suppose there is no net force on the particle. In the notation of §3, ${}_1\mathbf{P} = 0$ and to lowest order ${}_2\mathbf{P}$ is a symmetrical, periodic, second-order tensor with terms proportional to $a^3\kappa$. In terms of the dimensionless strained co-ordinate

$$\tilde{\mathbf{r}} = \kappa^{\frac{1}{2}}\mathbf{r}/\nu^{\frac{1}{2}}, \tag{B 2}$$

the equation satisfied by the outer expansion is

$$\tilde{\nabla} p' - \tilde{\nabla}^2 \mathbf{q}' + \tilde{z}\frac{\partial \mathbf{q}'}{\partial \tilde{x}} + q_3'\mathbf{e}_1 + \frac{1}{\kappa}\frac{\partial \mathbf{q}'}{\partial t} + \frac{1}{\kappa^{\frac{1}{2}}\nu^{\frac{1}{2}}}(\mathbf{q}'.\tilde{\nabla})\,\mathbf{q}' = \frac{\kappa}{\nu}\,{}_2\mathbf{P}.\tilde{\nabla}\delta(\tilde{r}) + \ldots. \tag{B 3}$$

Consider now the matching procedure. At any stage, the inner expansion is arbitrary to the extent of a solution of the homogeneous equation (B 1), with the right-hand side zero, which satisfies the boundary condition of zero velocity on the particle. It is almost obvious that the most important flow field that can give rise to a couple is one that has a constant rate of strain at infinity, and further that this flow field would be generated by matching with the terms in $\mathbf{q}'$ that are linearly proportional to $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ in the expansion about $\tilde{r} = 0$. We now argue that there are no terms of this type to order ν^{-1}.

The dominant term of order $\tilde{r}$ is proportional to the leading term on the right hand side of (B 3) and is therefore of the form

$$(\kappa/\nu)\,a^3\kappa\tilde{r}\mathbf{H}(\tilde{\mathbf{r}}, \kappa t) + o(\nu^{-1}), \tag{B 4}$$

where $\mathbf{H}$ is homogeneous of degree zero in $\tilde{x}$, $\tilde{y}$, $\tilde{z}$. In unstrained co-ordinates, (B 4) is proportional to $a^3\kappa^{\frac{5}{2}}r/\nu^{\frac{3}{2}}$, which is $O(\nu^{-\frac{3}{2}})$ and therefore makes no contribution to the terms in the inner expansion of order ν^{-1}. Terms in the inner expansion of higher than the first degree in the unstrained co-ordinates, which can also give rise to a couple, can readily be shown by a similar argument to be smaller than $O(\nu^{-\frac{3}{2}})$.

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